Introduction to Fourier Series

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1 Introduction

Different methods have been used to solve linear ordinary differential equations (ODEs) of the form $y'' + ay' + by = f(t)$, where a and $b \in \mathbb{R}$ and $f(t)$ was usually an exponential, trigonometric or polynomial function. There are, however, many functions, $f(t)$, for which the solution $y(t)$ cannot be directly solved using the methods learned thus far. An example of such functions is:

$$f(t) = \begin{cases} -1, & -\pi < t < 0; \\ 1, & 0 \leq t < \pi. \end{cases} \tag{1}$$

Functions such as (3), which are periodic and sometimes discontinuous, can however be written in a different form to ease the analysis; such function can be written in terms of their Fourier series, an infinite sum of sines and cosines:

$$f(t) = C_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt)\right), \text{where } n \text{ is the frequency.} \tag{2}$$

It is essential to see the importance of (2) when solving a linear ODE. Assuming $f_{n,s}(t) = \sin nt$ and $f_{n,c}(t) = \cos nt$ have solutions $y_{n,s}(t)$ and $y_{n,c}(t)$, respectively, a superimposed function $f(t) = C_0 + \sum (a_n f_{n,c}(t) + b_n f_{n,s}(t))$ would then have a solution of the form $y(t) = C_1 + \sum (a_n y_{n,c}(t) + b_n y_{n,s}(t))$, where C_0 and C_1 result from $n = 0$.

2 Finding the Fourier Series

Therorem 1. *Given the set of functions* $\mathbf{T}$ *as defined below:*

$$\mathbf{T} = \begin{cases} sin(nt), & n = 1, 2, \ldots, \infty; \\ cos(mt), & m = 0, 1, \ldots, \infty. \end{cases} \tag{3}$$

any two distinct functions, ϕ_n *and* $\phi_m \in \mathbf{T}$, *are orthogonal on the interval* $[-\pi, \pi]$, *such that* $n \neq m$.[1]

Proof. Two functions, $\phi_m(t)$ and $\phi_n(t)$, are said to be orthogonal on interval $[\alpha, \beta]$ if

$$\int_{\alpha}^{\beta} \phi_m(t)\phi_n(t)\, dt = 0, \text{where } m \neq n. \tag{4}$$

Given set $\mathbf{T}$ and the orthogonality condition (4), there are three different cases which must be calculated to cover all the permutations of the functions in $\mathbf{T}$.

[1] This paper uses the canonical form, so the interval is limited to $[-\pi, \pi]$ and the period of the functions is 2π. Different functions have different periods and different intervals for which the theorem holds.

Case I: Let $\phi_n = \sin(nt)$ and $\phi_m = \cos(mt)$ and $\alpha = -\pi$ and $\beta = \pi$

$$\int_{-\pi}^{\pi} \sin(nt)\cos(mt)\,dt = \begin{cases} 0, & m = n; \\ 0, & m \neq n. \end{cases}$$

Case II: Let $\phi_n = \cos(nt)$ and $\phi_m = \cos(mt)$ and $\alpha = -\pi$ and $\beta = \pi$

$$\int_{-\pi}^{\pi} \cos(nt)\cos(mt)\,dt = \begin{cases} \pi, & m = n; \\ 0, & m \neq n. \end{cases}$$

Case III: Let $\phi_n = \sin(nt)$ and $\phi_m = \sin(mt)$ and $\alpha = -\pi$ and $\beta = \pi$

$$\int_{-\pi}^{\pi} \sin(nt)\sin(mt)\,dt = \begin{cases} \pi, & m = n; \\ 0, & m \neq n. \end{cases}$$

All three cases show that for $m \neq n$, $\int_{-\pi}^{\pi} \phi_m(t)\phi_n(t)\,dt = 0$, proving theorem 1. □

2.1 Finding the Fourier Coefficients

From theorem 1 we can now determine a_n and b_n in (2), which is rewritten below for convenience:

$$f(t) = C_0 + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))$$

and is expanded to

$$f(t) = C_0 + \cdots + a_k \cos(kt) + \cdots + a_n \cos(nt) + \cdots + b_j \sin(jt) + \cdots + b_n \sin(nt)$$

which is further multiplying $f(t)$by $\cos(nt)$ and intergrated on the interval $[-\pi, \pi]$:

$$\begin{aligned} \int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \;=\; & \cdots + \overbrace{\int_{-\pi}^{\pi} a_k \cos(kt)\cos(nt)\,dt}^{0} + \cdots + \overbrace{\int_{-\pi}^{\pi} b_j \sin(jt)\cos(nt)\,dt}^{0} + \cdots \\ & \cdots + \underbrace{\int_{-\pi}^{\pi} a_n \cos^2(nt)\,dt}_{a_n\pi} + \cdots + \underbrace{\int_{-\pi}^{\pi} b_n \sin(nt)\cos(nt)\,dt}_{0} \end{aligned}$$

from which:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \tag{5}$$

Similarly multiplying $f(t)$ by $\sin(nt)$ and intergrating on the interval $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \cdots + \overbrace{\int_{-\pi}^{\pi} a_k \cos(kt)\sin(nt)\,dt}^{0} + \cdots + \overbrace{\int_{-\pi}^{\pi} b_j \sin(jt)\sin(nt)\,dt}^{0} + \cdots$$
$$\cdots + \underbrace{\int_{-\pi}^{\pi} a_n \cos(nt)\sin(nt)\,dt}_{0} + \cdots + \underbrace{\int_{-\pi}^{\pi} b_n \sin^2(nt)\,dt}_{b_n\pi}$$

from which:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt \tag{6}$$

Finally, C_0 is found by letting $n = 0$ to be:

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{a_0}{2}$$

Rewriting (2):

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nt) + b_n\sin(nt)\right), \tag{7}$$

where a_n and b_n are defined by (5) and (6), respectively.

2.2 General Form

The Fourier series of a more general periodic functions defined on the interval $(-p, p)$ is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi}{p}t\right) + b_n\sin\left(\frac{n\pi}{p}t\right)\right), \tag{8}$$

where:

$$a_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos\left(\frac{n\pi}{p}t\right)\,dt \qquad \text{and} \qquad b_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin\left(\frac{n\pi}{p}t\right)\,dt$$

3 Conclusion

It is important to see that as $N \to \infty$ the Fourier series of function $f(t)$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{N}\left(a_n\cos(nt) + b_n\sin(nt)\right)$$

converges to $f(t)$ on the defined interval. For functions with finite points of discontinuities, such as (2), the Fourier series converges to $f(t)$ at a point of continuity and $\frac{1}{2}(f(t+)+f(t-))$ at points of discontinuity [2]. It is also interesting to note that in many cases where $N \neq \infty$, but is still quite large, the Fourier series is a very close approximation and sufficient enough for many practical applications.

4 Examples

4.1 Example 1

Given $f(t) = 2e^2 - t, t \in (-\pi, \pi)$, find its Fourier series equivalent.

From (5):

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(2e^2 - t)\cos(nt)\,dt = \overbrace{\frac{1}{\pi}\int_{-\pi}^{\pi}2e^2\cos(nt)\,dt}^{0} - \frac{1}{\pi}\int_{-\pi}^{\pi}t\cos(nt)\,dt$$

Then letting $u = t$ and $dv = \cos(nt)\,dt$:

$$uv\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}v\,du = \frac{t\sin(nt)}{n}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin(nt)}{n}\,dt = \frac{t\sin(nt)}{n}\Big|_{-\pi}^{\pi} + \frac{\cos(nt)}{n^2}\Big|_{-\pi}^{\pi} = 0$$

from which

$$a_n = 0 - \frac{1}{\pi}\times 0 = 0$$

and

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\,dt = \frac{1}{\pi}\left(2e^2t - \frac{t^2}{2}\Big|_{-\pi}^{\pi}\right) = 4e^2$$

From (6):

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(2e^2 - t)\sin(nt)\,dt = \overbrace{\frac{1}{\pi}\int_{-\pi}^{\pi}2e^2\sin(nt)\,dt}^{0} - \frac{1}{\pi}\int_{-\pi}^{\pi}t\sin(nt)\,dt$$

Then letting $u = t$ and $dv = \sin(nt)\,dt$:

$$uv\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}v\,du = \frac{-t\cos(nt)}{n}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos(nt)}{n}\,dt = -\frac{t\cos(nt)}{n}\Big|_{-\pi}^{\pi} + \underbrace{\frac{\sin(nt)}{n^2}\Big|_{-\pi}^{\pi}}_{0}$$

from which

$$b_n = 0 - \frac{1}{\pi}\times\frac{-2\pi\cos(n\pi)}{n} = \frac{2\cos(n\pi)}{n} = \frac{2(-1)^n}{n}$$

and finally,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(b_n\sin(nt)) = 2e^2 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nt) \tag{9}$$

As previously mentioned in cases where $N \neq \infty$, the Fourier series of $f(t)$ is a close approximation and can still be of practical use. As Fig. 1 shows the Fourier series approximation for (9), with increasing N is very close to the actual function.

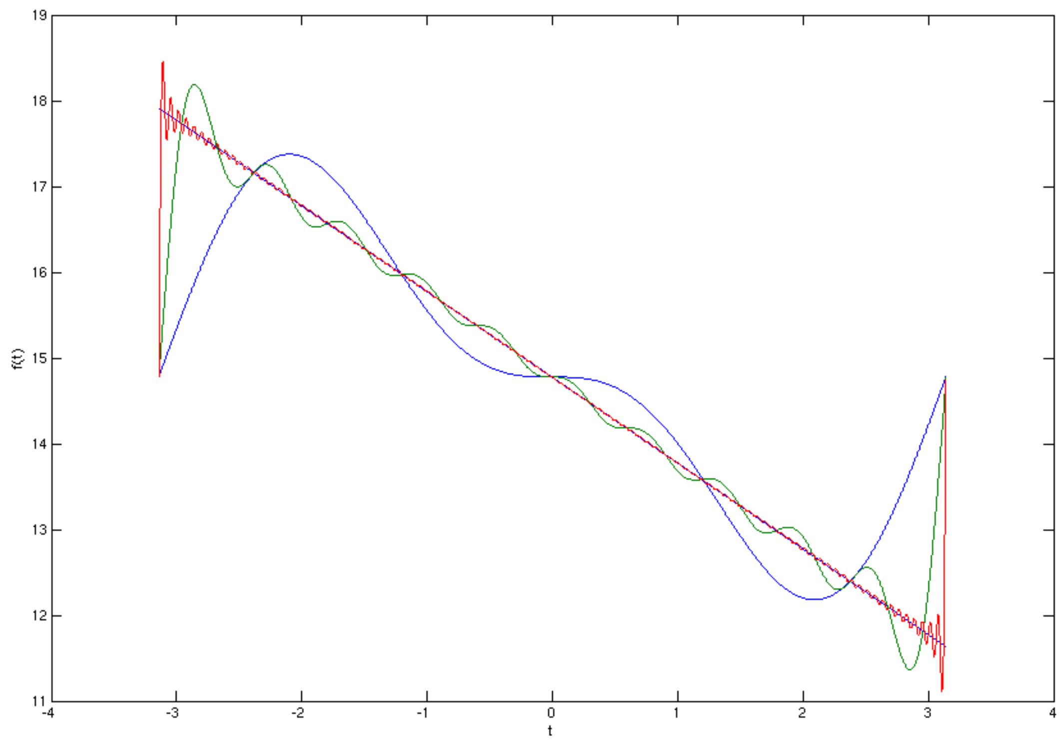

Figure 1: $f(t) = 2e^2 - t, t \in (-\pi, \pi)$ (black) and its Fourier series expanded with $N = 2(\text{blue}), 10(\text{green}), 100(\text{red})$.

4.2 Example 2

Given $f(t) = |t|, t \in (-\pi, \pi)$, find its Fourier series equivalent.

The function must first be rewritten as

$$f(t) = \begin{cases} -t, & -\pi < t < 0; \\ t, & 0 \leq t < \pi. \end{cases}$$

From (5):

$$\begin{aligned} a_n &= \tfrac{1}{\pi} \int_{-\pi}^{0} -t\cos(nt)\,dt + \tfrac{1}{\pi} \int_{0}^{\pi} t\cos(nt)\,dt \\ &= \tfrac{-1}{\pi} \left(\tfrac{\cos(nt)}{n^2} + \tfrac{t\sin(nt)}{n} \right) \Bigg|_{-\pi}^{0} + \tfrac{1}{\pi} \left(\tfrac{\cos(nt)}{n^2} + \tfrac{t\sin(nt)}{n} \right) \Bigg|_{0}^{\pi} \\ &= \tfrac{\cos(n\pi)-1}{\pi n^2} + \tfrac{\cos(n\pi)-1}{\pi n^2} = \tfrac{2(\cos(n\pi)-1)}{\pi n^2} \\ &= \tfrac{2((-1)^n-1)}{\pi n^2} \end{aligned}$$

and

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{0} -t\,dt + \frac{1}{\pi} \int_{-\pi}^{0} t\,dt = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Similarly, from (6):

$$\begin{aligned} b_n &= \tfrac{1}{\pi} \int_{-\pi}^{0} -t\sin(nt)\,dt + \tfrac{1}{\pi} \int_{0}^{\pi} t\sin(nt)\,dt \\ &= \tfrac{-1}{\pi} \left(\tfrac{\sin(nt)}{n^2} - \tfrac{t\cos(nt)}{n} \right) \Bigg|_{-\pi}^{0} + \tfrac{1}{\pi} \left(\tfrac{\sin(nt)}{n^2} - \tfrac{t\cos(nt)}{n} \right) \Bigg|_{0}^{\pi} \\ &= \tfrac{cos(n\pi)}{n} - \tfrac{cos(n\pi)}{n} \\ &= 0 \end{aligned}$$

finally,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt)) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos(nt) \tag{10}$$

The Fourier series of $f(t)$ provides an extremely close approximation of (10) and as shown in Fig. 2, even very small values, such as $N = 10$, provide a sufficiently close approximation.

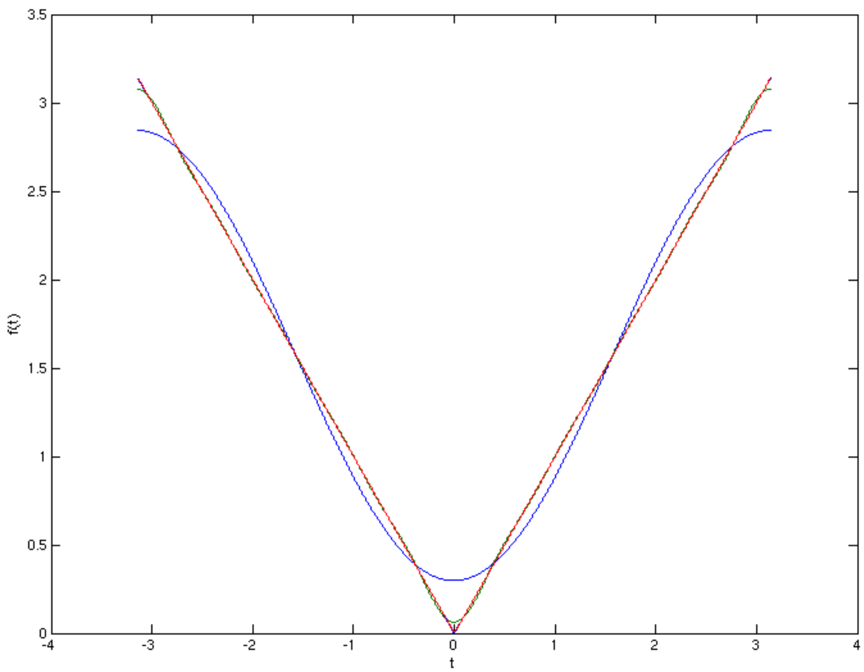

Figure 2: $f(t) = |t|, t \in (-\pi, \pi)$ (black) and its Fourier series expanded with $N = 2$(blue), 10(green), 100(red).

5 Acknowledgment

Much of this material is based on the 15th M.I.T. video lecture of Professor Arthur Mattuck's class 18.03 [1]. Much thanks to Prof. Mattuck and the M.I.T. OpenCourseWare group for making their lectures available.

6 Copyright

References

[1] Professor Arthur Mattuck. Introduction to fourier series; basic formulas for period 2(pi). Video lecture, 2003.

[2] Dennis G. Zill and Michael R. Cullen. *Differential Equations with Boundary-Value Problems*, chapter 11, page 437. Thomson Brooks/Cole, 6th edition, 2005.